*Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 2, pp. 204–208, 2004*

# NONUNIQUENESS OF CONJUGATE FLOWS

**N. I. Makarenko**

UDC 532.592

*The problem of weakly stratified flows conjugate to a uniform flow with a prescribed density distribution over its depth is considered. The sufficient condition of existence and uniqueness of the conjugate flow is obtained for a smooth generic background density profile. If the condition obtained is violated, it is shown that the number of branches of conjugate flows and their asymptotic behavior near the bifurcation point are determined by the fine structure of stratification.*

**Key words:** *stratification, conjugate flows.*

**Introduction.** Two steady horizontal flows of a nonuniform fluid are called conjugate if they are matched with each other in terms of conservation laws [1]. Flows of this kind with identical fluxes of mass, momentum, and energy are formed in wave configurations of the smooth bore and also in plateau-shaped solitary internal waves with flat crests. The search for pairs of conjugate motions reduces to a one-dimensional (in terms of spatial variables) bifurcation problem whose solutions branch off in eigenvalues of the linearized problem corresponding to the countable family of internal wave modes on the upstream flow. The nonuniqueness is understood as a situation where a fixed mode generates more than one branch of conjugate flows. Such nonuniqueness leading to interesting bifurcations of wave structures was noted by a number of authors (see, e.g., [2–4]) in numerical calculations for continuous stratification with two pycnoclines and its simplified three-layer model. Nonuniqueness of conjugate flows was established analytically in [5] for stratification close to linear. In the present work, an attempt is made to characterize the conditions on the basic-flow density profile, which are responsible for the uniqueness and nonuniqueness properties.

**1. Formulation of the Problem.** The equations of steady motions of a nonuniform fluid have the form

$$\rho(UU_x + VU_y) + p_x = 0, \qquad \rho(UV_x + VV_y) + p_y = -\rho g,$$
$$U_x + V_y = 0, \qquad U\rho_x + V\rho_y = 0, \tag{1.1}$$

where $\rho$ is the fluid density, $U$ and $V$ are the velocity-vector components, $p$ is the pressure, and $g$ is the acceleration of gravity. We consider the motions in the layer $-\infty < x < +\infty$, $0 < y < h$ between a flat bottom $y = 0$ and a rigid cover $y = h$. Conservation of density along the streamlines implies a functional dependence of density $\rho = \rho(\psi)$ of the stream function $\psi$ determined by the relations $U = \psi_y$ and $V = -\psi_x$. With allowance for this dependence, elimination of pressure by virtue of the Bernoulli integral

$$\rho|\nabla\psi|^2/2 + \rho g y + p = H(\psi)$$

reduces system (1.1) to the second-order differential equation — quasilinear elliptic Dubreil-Jacotin–Long equation

$$\rho'(\psi)\,\Delta\psi + \rho(\psi)(gy + |\nabla\psi|^2/2) = H'(\psi),$$

where the function $H(\psi)$ is determined by the basic-flow parameters. The problem of flows conjugate to a uniform flow $\psi = cy$ with the density law $\rho = \rho_\infty(y)$ and the corresponding hydrostatic pressure $p = p_\infty(y)$ is formulated as a nonlinear problem on eigenvalues for the one-dimensional Dubreil-Jacotin–Long operator

$$\rho(\psi)\psi_{yy} + \rho'(\psi)(gy - g\psi/c + (\psi_y^2 - c^2)/2) = 0; \tag{1.2}$$

---

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 2, pp. 68–74, March–April, 2004. Original article submitted October 27, 2003.

 0021-8944/04/4502-0204 

$$\psi = 0 \quad (y = 0), \qquad \psi = ch \quad (y = h), \tag{1.3}$$

where $\rho(\psi) = \rho_\infty(\psi/c)$. Here, the basic-flow velocity $c > 0$ is a spectral parameter. We seek flows without the reverse flow of the fluid, i.e., $\psi_y > 0$ for $0 < y < h$. Setting $\rho$ and $H$ as functions of $\psi$, taken into account in the structure of Eq. (1.2), automatically involves conservation of mass and energy in the sought flows. In contrast, conservation of the total horizontal momentum

$$\int_0^h (p + \rho\,\psi_y^2)\,dy = \int_0^h (p_\infty + \rho_\infty c^2)\,dy$$

is an additional condition, which should be considered together with Eqs. (1.2) and (1.3).

We choose the quantities $h/\pi$, $ch/\pi$, and $\rho_\infty(0)$ as scales for $y$, $\psi$, and $\rho$, respectively. The motion is described by two dimensionless constants: Boussinesq parameter $\sigma$ and parameter $\lambda$, which is a squared inverse densimetric Froude number

$$\sigma = \frac{N_0^2 h}{\pi g}, \qquad \lambda = \frac{\sigma g h}{\pi c^2}.$$

Here $N_0$ is the characteristics buoyancy frequency $N$ in the basic flow; $N^2(y) = -g\rho'_\infty(y)/\rho_\infty(y)$. In dimensionless variables, Eqs. (1.2) and (1.3) for the disturbances of the uniform flow stream function $v = \psi(y) - y$ acquire the form

$$F(v;\sigma,\lambda) \equiv (\rho v_y)_y - \rho_\psi(\sigma^{-1}\lambda v + v_y^2/2) = 0, \qquad v(0) = v(\pi) = 0, \tag{1.4}$$

where $\rho = \rho(y + v, \sigma)$. The problem considered admits a variational formulation according to which the integral of the momentum flux after pressure elimination can be written as

$$\int_0^\pi L\,dy = 0, \tag{1.5}$$

where $L(v;\sigma,\lambda)$ is the Lagrangian of the operator $F = \delta L/\delta v$,

$$L = -\frac{1}{2}\,\rho(y+v,\sigma)v_y^2 + \sigma^{-1}\lambda \int_y^{y+v} (\rho(\psi,\sigma) - \rho(y+v,\sigma))\,d\psi.$$

In other words, conjugate flows with the condition of conservation of mass, energy, and total horizontal momentum are critical points of functional (1.5) on its zero-level surface.

In the case of weak stratification, the Boussinesq parameter is a natural small parameter in problem (1.4), (1.5). In accordance with the known concepts of the properties of thermohaline stratification of sea water [6], the density distribution can be described by the equation

$$\rho(\psi,\sigma) = 1 - \sigma\rho_*(\psi) - \sigma^2\rho_1(\psi,\sigma),$$

where the coefficient $\rho_*$ defines the background profile, and the function $\rho_1$ describes the fine structure of the density field. Stratification is assumed to be stable; hence, the functions $\rho_* \in C^4[0,\pi]$ and $\rho_1 \in C^4([0,\pi]\times[0,\sigma_0])$ satisfy the inequalities $\rho > 0$, $\rho_{*\psi} > 0$, and $\rho_\psi < 0$ for $\psi \in [0,\pi]$ and $\sigma \in [0,\sigma_0]$ with a certain $\sigma_0 > 0$.

Under natural conditions, there is a comparatively small number of functional dependences typical for the mean density profile $\rho_*$. In particular, they include the linear and exponential dependences of density on depth, stratification with one or several pycnoclines, and combinations of the above-mentioned profiles. Conversely, the fine structure of stratification is much more versatile and changeable under the action of such factors as daily heating and cooling of water, salt diffusion, breaking of internal waves, etc. Nevertheless, the characteristic time of its evolution is noticeably greater than the time periods of internal waves; therefore, in modeling wave processes, it can also be defined by a steady dependence $\rho_1$, for which there are fairly reliable methods of probing.

**2. Sufficient Condition of Existence and Uniqueness.** The study of solvability of problem (1.4), (1.5) is based on its reduction to an equivalent system of two implicitly set scalar equations for three parameters: amplitude, Froude number, and Boussinesq parameter. Constructing of branching equations involves the functional spaces

$$X = \{v \in C^2[0,\pi]\colon v(0) = v(\pi) = 0\}, \qquad Y = C[0,\pi].$$

Let $B_r = \{v \in X\colon \|v\|_X < r\}$ be a sphere of radius $r > 0$ in the space $X$. Note, for all $v \in B_r$ with sufficiently small $r$, the stream function $\psi = y + v(y)$ takes the values in the domain of definition $0 \leqslant \psi \leqslant \pi$ of density $\rho$. Hence, the mapping $F\colon B_r \times [0, \sigma_0] \times \mathbb{R} \to Y$ is determined correctly and is smooth. The Sturm–Liouville problem

$$F'_v(0; 0, \lambda)\varphi \equiv \varphi_{yy} + \lambda\rho'_*(y)\varphi = 0, \qquad \varphi(0) = \varphi(\pi) = 0 \tag{2.1}$$

defines, in the Boussinesq approximation $\sigma = 0$, the spectrum of normal modes of the uniform flow $\psi = y$, consisting of simple eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$, where $\lambda_n \to +\infty$ $(n \to \infty)$. Let $\varphi_n \in X$ be eigenfunctions orthonormalized in $L_2[0, \pi]$, and let $Q_n v = \varphi_n (v, \varphi_n)_{L_2[0,\pi]}$ be a projector onto the corresponding one-dimensional subspace. Branching of small solutions of problem (1.4) from the zero solution can occur only in eigenvalues $\lambda_n$. In accordance with the Lyapunov–Schmidt method, the equation for the function $v$

$$B_n v = R(v; \sigma, \lambda)$$

with the Fredholm operator $B_n = F'_v(0; 0, \lambda_n)$, which has a one-dimensional kernel in $X$ and a one-dimensional co-kernel in $Y$, and the small operator $R$ for which $R(0; 0, \lambda_n) = 0$, $R'_v(0; 0, \lambda_n) = 0$, is equivalent to the scalar branching equation

$$f(b, \sigma, \lambda) \equiv \int_0^\pi \varphi_n(y) F(b\varphi_n(y) + w_n(y; b, \sigma, \lambda); \sigma, \lambda)\, dy = 0. \tag{2.2}$$

The mapping $(b, \sigma, \lambda) \to w_n(y; b, \sigma, \lambda) \in (I - Q_n)X$ of class $C^2$ involved here is uniquely determined in accordance with the implicit function theorem as the solution of the operator equation

$$B_n w = (I - Q_n) R(b\varphi_n + w; \sigma, \lambda).$$

With this mapping $w_n$, relation (1.5) yields one more scalar equation for the parameters $b$, $\sigma$, and $\lambda$:

$$l(b, \sigma, \lambda) \equiv \int_0^\pi L(b\varphi_n(y) + w_n(y; b, \sigma, \lambda); \sigma, \lambda))\, dy = 0. \tag{2.3}$$

Thus, the search for conjugate flows of the $n$th mode close to a uniform flow is equivalent to the search for the solutions $(b, \sigma, \lambda)$ of system (2.2), (2.3) in a small neighborhood of the point $(0, 0, \lambda_n)$.

We calculate the coefficients of the system of branching equations at the leading order. To separate the branch of trivial solutions, we preliminary note that the element $w_n$ orthogonal to the eigenfunction $\varphi_n$ admits the estimates

$$\|w_n(\,\cdot\,; b, \sigma, \lambda)\|_X \leqslant C|b|, \qquad \|w_n(\,\cdot\,; b, 0, \lambda)\|_X \leqslant Cb^2,$$

which are uniform with respect to $\sigma$ and $\lambda$ near the critical point $(\sigma, \lambda) = (0, \lambda_n)$. These inequalities immediately follow from the structure of the operator $R$. According to this comment, the dependence of the functions $f$ and $l$ on the amplitude parameter $b$ inherit the asymptotic orders of the operator $F$ and Lagrangian $L$ for small $v$, admitting a representation in the form

$$f(b, \sigma, \lambda) = b f_1(b, \sigma, \lambda), \qquad l(b, \sigma, \lambda) = b^2 l_1(b, \sigma, \lambda)$$

with smooth functions $f_1$ and $l_1$. Separating the branch of solutions $b = 0$ corresponding to a uniform flow, we present system (2.2), (2.3) in the form

$$A_n \boldsymbol{a} = \boldsymbol{f}(\boldsymbol{a}; \sigma), \qquad \boldsymbol{a} = (b, \lambda - \lambda_n),$$

where

$$A_n = \frac{\partial(f_1, l_1)}{\partial(b, \lambda)}\Big|_{b=0,\, \sigma=0,\, \lambda=\lambda_n},$$

and the nonlinear right side $\boldsymbol{f}$ is such that $\boldsymbol{f}(0; 0) = 0$. We find the coefficients of the matrix $A_n$. For the operator $F(v; 0, \lambda_n) = v_{yy} + \lambda_n \rho'_*(y + v)v$, we have

$$\lim_{b \to 0} b^{-2} F(b\varphi_n + w_n(\cdot\,; b, 0, \lambda_n); 0, \lambda_n) = \psi_{nyy} + \lambda_n \rho'_*(y)\psi_n + \lambda_n \rho''_*(y)\varphi_n^2$$

with the designation $\psi_n(y) = (1/2) w_{nbb}(y; 0, 0, \lambda_n)$. Since the element $B_n \psi_n = \psi_{nyy} + \lambda_n \rho'_*(y)\psi_n$ is orthogonal in $L_2[0, \pi]$ to the eigenfunction $\varphi_n$, we obtain

$$f_{1b}(0,0,\lambda_n) = \lambda_n \int\limits_0^\pi \rho_*''(y)\varphi_n^3(y)\,dy.$$

Since the operator $F(v;\sigma,\lambda)$ depends linearly on $\lambda$ and $F_\lambda(v;0,\lambda) = \rho_*'(y+v)v$, we find

$$f_{1\lambda}(0,0,\lambda_n) = \int\limits_0^\pi \rho_*'(y)\varphi_n^2(y)\,dy.$$

To calculate the derivative $l_{1\lambda}$, it suffices to use the expression

$$L_\lambda(v;0,\lambda) = \int\limits_y^{y+v} \Big(\rho_*(y+v,\sigma) - \rho_*(\psi,\sigma)\Big)\,d\psi,$$

which, by virtue of the variational property of the initial problem, yields the relation $l_{1\lambda}(0,0,\lambda_n) = (1/2)f_{1\lambda}(0,0,\lambda_n)$. Similarly, we find the coefficient $l_{1b}(0,0,\lambda_n) = (1/3)f_{1b}(0,0,\lambda_n)$. Thus, the matrix $A_n$ has the determinant

$$\det A_n = \frac{1}{6}\,\lambda_n \int\limits_0^\pi \rho_*'(y)\varphi_n^2(y)\,dy \times \int\limits_0^\pi \rho_*''(y)\varphi_n^3(y)\,dy.$$

On the basis of the implicit function theorem, we obtain the following statement.

**Theorem 1.** *If* $\det A_n \neq 0$*, there is the unique branch of conjugate flows close to the basic flow for the mode with the number* $n$*:*

$$\psi(y;\sigma) = y + b(\sigma)\,\varphi_n(y) + O(b^2), \quad b(\sigma)\to 0, \quad \lambda(\sigma)\to\lambda_n \qquad (\sigma\to 0).$$

For density distributions satisfying the condition of this theorem, the fine structure of stratification does not exert any effect on determining the number of families of small solutions of problem (1.4), (1.5). Such profiles are similar to two-layer stratification with constant densities of the fluid in the layers, for which there is only one branch of piecewise-constant flows conjugate to a uniform flow. In particular, according to Theorem 1, local uniqueness always occurs in conjugate of the first mode in the case of a convex profile $\rho^*$, since the eigenfunction $\varphi_1(y)$ has no zeros at the points of the interval $y\in[0,\pi]$ other than the end points.

**3. Example of Nonuniqueness.** The condition of Theorem 1 is certainly invalid for the density with a linear function $\rho_*$. This interesting case includes both the linear stratification with $\rho_1 = 0$ and the exponential stratification $\rho = \exp(-\sigma y)$ with small $\sigma$. In the case considered, the limiting problem (1.4) with $\sigma = 0$ is linear; its eigenfunctions and eigenvalues are found in an explicit form:

$$\varphi_n(y) = \sqrt{2/\pi}\,\sin ny, \qquad \lambda_n = n^2.$$

This allows us to consider large-amplitude conjugate flows with local parameters strongly different from the local characteristics of the basic flow. The only restriction here is the requirement of the absence of reverse flows, which is satisfied under the condition on the amplitude $|b| < 1/n$ for the mode with the number $n$. In the case considered, the system of branching equations can be conveniently represented in a somewhat different form, identifying the linear part with respect to the Boussinesq parameter $\sigma$ and the number $\lambda$ after separation of the branch of trivial solutions:

$$W_n(b)\boldsymbol{a} = f(\boldsymbol{a};b), \qquad \boldsymbol{a} = (\sigma,\lambda-\lambda_n).$$

A significant fact is that the matrix $W_n$, by virtue of the variational property of the initial problem, has the structure of the Wronskian

$$W_n(b) = \begin{pmatrix} s_n(b) & m_n(b) \\ s_n'(b) & m_n'(b) \end{pmatrix},$$

where the coefficient $m_n(b) = b^2/2$ is the same for all modes, and the dependence of the coefficient $s_n(b)$ on the amplitude $b$ is determined only by the function $\rho_1$ describing the fine structure of stratification:

$$s_n(b) = \frac{2n^2}{\pi}\int\limits_0^\pi \int\limits_y^{y+b\sin ny} \Big(\rho_1(y+b\sin ny,0) - \rho_1(\psi,0)\Big)\,d\psi\,dy + \frac{(\pi nb)^2}{4} + \frac{n}{6}\,(1-(-1)^n)b^3.$$

A sufficient condition for the existence of conjugate flows was obtained in [5]; like Theorem 1, this condition is formulated in terms of the determinant of the matrix of the linear part of the system of branching equations. Namely, let $b_0 \neq 0$ be a simple root of the function $\det W_n(b)$ in the interval $(-1/n, 1/n)$; then, for a particular $b_0$, there exists a unique branch of conjugate flows for which $(v(y;\sigma), \lambda(\sigma)) \to (b_0 \sin ny, n^2)$ as $\sigma \to 0$, and the following asymptotic solution is valid:

$$\lambda(\sigma) = n^2 - 2s_n(b_0)\sigma/b_0^2 + O(\sigma^2). \tag{3.1}$$

Based on this condition, we give a formulation that offers a simple analytical interpretation of the nonuniqueness property and simultaneously allows classification of sub- and supercritical conjugate flows.

**Theorem 2.** *Let* $b_0 \in (-1, 1)$, $b_0 \neq 0$ *by the point of a local extremum of the function* $\Lambda_1(b) = -2s_1(b)/b^2$, *and* $\Lambda_1''(b_0) \neq 0$. *Then, this point corresponds to a conjugate flow of the first mode, which is supercritical for* $\Lambda_1(b_0) < \Lambda_1(0)$ *and subscritical for* $\Lambda_1(b_0) > \Lambda_1(0)$.

Indeed, the existence of the corresponding branch follows from the above-mentioned sufficient condition, since the extremum points indicated in the theorem are simple zeros for the function $\det W_1(b) = (1/4)b^4\Lambda_1'(b)$. According to the definition of the densimetric Froude number, the conjugate flow is supercritical is $\lambda(\sigma) < \lambda_c(\sigma)$ and subscritical for $\lambda(\sigma) > \lambda_c(\sigma)$, where the critical value $\lambda_c(\sigma)$ is the first eigenvalue of the Sturm–Liouville problem

$$(\rho(y,\sigma)\varphi_y)_y - \lambda\sigma^{-1}\rho_y(y,\sigma)\varphi = 0, \qquad \varphi(0) = \varphi(\pi) = 0,$$

which yields the long-wave spectrum of normal modes of the uniform flow. As was already noted, in the Boussinesq approximation $\sigma = 0$, this problem has the form (2.1). Therefore, for stratification close to linear, the first eigenvalue for small $\sigma$ has the asymptotic solution $\lambda_1(\sigma) = 1 + \Lambda_1(0)\,\sigma + O(\sigma^2)$. A comparison with the asymptotic branch of the conjugate flows (3.1) yields the statement of Theorem 2.

In dimensionless variables, formula (3.1) establishes the functional relation between the basicflow velocity, characteristic gradient of density of the liquid, and amplitude of the conjugate flow. The dependence of velocity on amplitude is determined by the above-introduced function $\Lambda_n(b) = -2s_n(b)/b^2$. According to Theorem 2, nonuniqueness of conjugate flows of a fixed mode occurs if this function on the amplitude parameter is nonmonotonic.

It is of interest to note that exponential and linear stratifications themselves do not offer examples of nonuniqueness (in this case, there are no nonzero points of the extremum of the function $\Lambda_n$), but density disturbances of order $O(\sigma^2)$ can generate an arbitrary number of branches prescribed beforehand. Thus, in the case of a linear background profile, the fine structure of stratification plays the governing role in the description of conjugate flows and associated nonlinear wave structures.

The author is grateful to A. Yu. Kazakov for useful comments.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 03-01-00829) within the framework of the Integrated Project No. 131 of the Siberian Division of the Russian Academy of Sciences and the program "Leading Scientific Schools" (Grant No. NSh 440.2003.1).

## REFERENCES

1. T. B. Benjamin, "A unified theory of conjugate flows," *Philos. Trans. Roy. Soc. London, Ser. A*, **269**, 587–643 (1971).
2. K. G. Lamb and B. Wan, "Conjugate flows and flat solitary waves for a continuously stratified fluid," *Phys. Fluids*, **10**, No. 8, 2061–2079 (1998).
3. K. G. Lamb, "Conjugate flows for a three-layer fluid," *Phys. Fluids*, **12**, No. 9, 2169–2185 (2000).
4. P.-O. Ruså̊s and J. Grue, "Solitary waves and conjugate flows in a three-layer fluid," *Europ. J. Mech., B. Fluids*, **21**, No. 2, 185–206 (2002).
5. N. I. Makarenko, "Conjugate flows and smooth bores in a weakly stratified fluid," *J. Appl. Mech. Tech. Phys.*, **40**, No. 2, 249–257 (1999).
6. K. N. Fedorov, *Fine Thermohaline Structure of Ocean Waters* [in Russian], Gidrometeoizdat, Leningrad (1976).